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S-2633

M. Sc. I (Sem. I) (Mathematics) Examination

March / April – 2011

Paper - 402 : Complex Analysis

Time : **3** Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :
☛ **M. SC. 1 (SEM. 1) (MATHEMATICS)**

Name of the Subject :
☛ **402 : COMPLEX ANALYSIS**

☛ Subject Code No. : **2 6 3 3** ☛ Section No. (1, 2,.....) : **NIL**

Seat No. :

Student's Signature

(2) This paper contains **five** questions.
(3) Follow **usual** notations and conventions.
(4) Figures on **right** indicates marks for that section.

1 (a) Define the following terms : **6**

(i) Analytic function
(ii) Harmonic function.
Derive the Cauchy - Riemann differential equation for a complex analytic function.

$$w = f(z) = u(x, y) + i\, v(x, y)$$

(b) Attempt any **two** : **8**

(i) Show that $u = x^3 - 3xy^2$ is harmonic and find a corresponding analytic function

$$f(z) = u(x, y) + i\ v(x, y)$$

(ii) Find all solutions of the equation $z^4 = 8\sqrt{2}\,(1+i)$

Leave your answers in exponential form and plot them on an Argand diagram.

(iii) Use the Cauchy - Riemann differential equations to show that $f(z) = \dfrac{1}{z^4}$ is analytic.

2 (a) Define the term : **6**

Radius of convergence of the power series.

Show that the radius of convergence of the Binomial series $\sum_{0}^{\infty}\binom{\alpha}{n}t^n$ is ≥ 1; where α is any nonzero complex number.

(b) Attempt any **two** : **8**

(i) Show that for arg $r > 0$, the series $\sum \frac{z^n}{n!}$ converges absolutely and uniformly for $|z| \leq r$.

(ii) Let $\{a_n\}$ be the sequence of numbers defined by $a_0 = 1,\ a_1 = 2$ and $a_n = a_{n-1} + a_{n-2}$; $n \geq 2$

Find the radius of convergence of the power series.

(iii) Define the term : Uniformly convergent sequence. If a sequence $\{f_n\}$ of functions on a set S is Cauchy, then show that it converges uniformly.

3 (a) Define : Holomorphic function. Let R be a rectangle, and f be a holomorphic function on R. Show that $\int_{\partial R} f = 0$, where ∂R is the boundary of the rectangle. **6**

(b) Attempt any **two** : **8**

(i) Evaluate $\int_C \frac{z^2 - \frac{1}{3}}{z^3 - z} dz$,

where , $C : \left| z - \frac{1}{2} \right| = 1$ with counter - clockwise orientation.

(ii) Use Cauchy - Integral formula

$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i \cdot f(z_0)$ to integrate $\int_C \frac{z^2+1}{z^2-1}\,dz,$

where C : is a circle of radius 1 with centre at z_0; in the counter - clockwise direction, where

(1) $z_0 = 1$

(2) $z_0 = i$

(iii) Let S be a convex set, and let γ, η be continuous closed curves in S. Show that γ, η are homotopic in S.

4 (a) Let $\overline{D}$ be a closed disc of positive radius and let f be halomorphic on an open disc U containing $\overline{D}$. Let V be the circle in which is the boundary of $\overline{D}$. Show that **6**

for every $z_0 \in \overline{D}$, $f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\,dz$.

(b) Attempt any **two** : **8**

(i) Show that $f(z) = \sum_1^{\infty} \frac{1}{(a+n)^z}$ is holomorphic for $R_e(z) > 1$; where $a > 0$.

(ii) Find the poles for the function $f(z) = \frac{2z}{(z+4)(z-1)^2}$.

Determine the order of the poles. Further, calculate the residues at each of the poles.

(iii) Define : Winding number $W(v, \alpha)$ for a closed path γ and with respect to a point α.

5 (a) Let $Q(x, y)$ be a rational function which is continuous when $x^2 + y^2 = 1$. **6**

Show that $\int_0^{2\pi} Q(\cos\theta, \sin\theta)\, d\theta = 2\pi i$

(Σ Residues of f inside the unit circle)

$f(z) = \frac{1}{iz} Q(\cos\theta, \sin\theta)$,

Where the function F is defined by

$z = e^{i\theta} = x + iy$.

Hence, show that $\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos\theta}\, d\theta = \frac{2\pi}{1 - a^2}$; for $0 < a < 1$.

(b) Attempt any **two** : **8**

(i) Find the Laurent's series of the function

$f(z) = \frac{1}{z^2(1-z)}$ in the regions

(1) $0 < |z| < 1$ and

(2) $|z| > 1$

(ii) Let γ, η be two continuous paths in an open set U and suppose they have the same starting and ending points. Further, assume that they are close together. Let f be holomorphic on U.

Show that $\int_\gamma f = \int_\eta f$

(iii) Let g have a simple pole at 0. Then show that

$\lim_{\in \to 0} \int_{S(\in)} g(z)\,dz = \pi i \,\mathrm{Res}_0(g)$. Use the above result

to prove that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.